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$(0, 2)$ Pál-type Interpolation: A General Method for Regularity

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Abstract. The methods of proof of regularity for interpolation problems often are dependent on the problem at hand. In case of given pairs of node generating polynomials the method of deriving an ordinary differential equation for the interpolating polynomial or that of exploiting the specific form of the node generator have mainly been used up to now.
Recently another method was used in the case of Pál-type interpolation where 'only' one of the node generators is fixed in advance: a 'general' method of deriving a companion generator that leads to a regular interpolation problem. Using $(0, 2)$ Pál-type interpolation, it is shown that each of the methods has its merits and for sake of simplicity we will restrict ourselves to the case that the nodes are the zeros of pairs of polynomials of the following form: $\{p(z)q(z), p(z)\}$ with p, q co-prime and both having simple zeros.

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1. Introduction

The study of Hermite-Birkhoff interpolation is a well-known subject (cf. the excellent book [2]). Recently the regularity of some interpolation problems on non-uniformly distributed nodes on the unit circle has been studied.

Along with the continuing interest in interpolation in general, a number of papers on Pál-type interpolation have appeared, cf. [3], [4], [6].

In this paper the attention will be focused on so-called $(0, 2)$ Pál-type interpolation problems on the pair of node generators $\{p(z)q(z), p(z)\}$:

- given two co-prime polynomials $p(z)$ resp. $q(z)$, with simple zeros $\{z_i\}_{i=1}^{n} \in \mathbb{C}$ resp. $\{w_j\}_{j=1}^{m} \in \mathbb{C}$ (*nodes generators*),
- given *data* $\{c_i\}_{i=1}^{n+m}$, $\{d_j\}_{j=1}^{n} \in \mathbb{C}$,

find $P_k \in \Pi_k,\ k = m + 2n - 1$ with $P_k(z_i) = c_i$ $(1 \le i \le n)$, $P_k(w_j) = c_{n+j}$ $(1 \le j \le m)$ and $P_k''(z_i) = d_i$ $(1 \le i \le n)$.

Here Π_k is the set of polynomials of degree at most k with complex coefficients. This type of interpolation problems started with the paper [1] by L.G. Pál in 1975.

Although very often the method of proof of regularity depends on the problem at hand, one can, nevertheless, distinguish two main tools as indicated in [5]:

A. Prove that the square system of homogeneous linear equations for the unknown coefficients of the polynomial P_k has a non-vanishing determinant.

B. Find a differential equation for P_k (or for a factor of P_k) and show that if this equation has a polynomial solution, the solution must be the trivial one.

Recently a new method has been introduced by the authors in [7] for $(0,1)$ Pál-type interpolation:

C. Given $p(z)$ 'only', apply a 'reduction method' and determine 'companion polynomial(s)' $q(z)$ that make the problem regular.

The layout of the paper is as follows: in section 2 general results for method **C** will be given, followed in section 3 by new results on $(0,2)$ Pál-type interpolation. In section 4 the general theorems from section 2 will be proved and in section 5 the proofs for the new examples will be given, using each of the methods **A**, **B** and **C**, along with a discussion of the relative merits of the three methods. Finally some references will be given.

2. General results for method C

Consider the node-generating polynomials

$$p(z) = \prod_{i=1}^{n} (z - z_i) \tag{1}$$

and

$$q(z) = \prod_{j=1}^{m} (z - w_j), \tag{2}$$

co-prime and each having simple zeros.

Remark. It is *not* allowed that p and q have (a) common zero(es).

We then have the following result

Theorem 2.1. *If there exist polynomials $g(z)$, $r_1(z)$, $r_2(z)$ such that*

$$2p'(z)q(z) = (\alpha_0 + \alpha_1 z)g(z) + r_1(z)p(z), \tag{3a}$$

$$p''(z)q(z) + 2p'(z)q'(z) = \beta_0 g(z) + r_2(z)p(z), \tag{3b}$$

satisfying the condition

$$\alpha_0 + \alpha_1 z \not\equiv 0,\ \ g(z_i) \neq 0,\ \ 1 \leq i \leq n, \tag{4}$$

then $(0,2)$ Pál-type interpolation on the zeros of $\{p(z)q(z), p(z)\}$ is regular

1. *for $\alpha_1 = 0$ if and only if $\beta_0 \neq 0$,*
2. *for $\alpha_1 \neq 0$ if and only if $-\beta_0/\alpha_1 \notin \{0, 1, 2, \ldots, n-1\}$.*

Remark. The case $\alpha_0 = \alpha_1 = 0$ leads to a contradiction with (3a) as the zeros of p are simple and the polynomials p, q are co-prime.

More general, using simple conditions on the factors of $g(z)$ from (5a), (5b):

Theorem 2.2. *If there exist polynomials $g(z)$, $r_1(z)$, $r_2(z)$ such that*

$$2p'(z)q(z) = (\alpha_0 + \alpha_1 z + \alpha_2 z^2)g(z) + r_1(z)p(z), \tag{5a}$$

with $\alpha_0 + \alpha_1 z + \alpha_2 z^2$ having two different (complex) roots σ_1, σ_2, and

$$p''(z)q(z) + 2p'(z)q'(z) = (\beta_0 + \beta_1 z)g(z) + r_2(z)p(z), \tag{5b}$$

satisfying the conditions

$$g(z_i) \neq 0,\ 1 \leq i \leq n, \tag{6}$$

and

$$A := \frac{\beta_0 + \beta_1\sigma_1}{\sigma_1 - \sigma_2} > 0,\ B := \frac{\beta_0 + \beta_1\sigma_2}{\sigma_2 - \sigma_1} > 0, \tag{7}$$

$$\int_{\sigma_1}^{\sigma_2} (\zeta - \sigma_1)^{A-1}(\zeta - \sigma_2)^{B-1} p(\zeta) d\zeta \neq 0, \tag{8}$$

then the $(0, 2)$ Pál-type interpolation problem on the zeros of $\{p(z)q(z), p(z)\}$ is regular.

3. New regular problems

In this section some new results on regularity are given.

Theorem 3.1. *The $(0, 2)$ Pál-type interpolation problem on the zeros of the pair $\{p(z)q(z), p(z)\}$, with p, q co-prime and having simple zeros, is regular for the following choices of the node generators:*

1. $p(z) = z^n - \alpha^n$, $\alpha \neq 0$; $q(z) = z$, $n \geq 1$.

2. $p(z) = z^n - \alpha^n$, $\alpha \neq 0$; $q(z) = z^n - \beta^n$, $\beta \neq 0$ *and*

$$\begin{cases} \alpha^n \neq \beta^n & \text{for } n = 1, \\ \alpha^n \neq \beta^n,\ (3n + 2k - 1)\alpha^n \neq (n + 2k - 1)\beta^n & \text{for } n \geq 2. \end{cases}$$

3. $p(z) = z^n - \alpha^n$, $q(z) = (z^{kn} - \alpha^{kn})/(z^n - \alpha^n)$, $\alpha \neq 0$, $k \geq 2$.

4. $p(z) = z^n - \alpha^n$, $q(z) = z(z - z_0)(z^n - \frac{3n+1}{n+1}\alpha^n)$ *with*

$$\alpha,\ z_0 \neq 0;\ z_0 \neq \alpha \exp\left(\frac{2\pi i k}{n}\right),\ k = 0, 1, \ldots, n-1;\ z_0^n \neq \frac{3n+1}{n+1}\alpha^n.$$

5. $p(z) = z^n - \alpha^n$, $q(z) = z(z^2 - \xi^2)(z^n - \frac{3n+1}{n+1}\alpha^n)$ *with*

$$\alpha,\ \xi \neq 0;\ \xi^2 \neq \alpha^2;\ \begin{cases} \xi^n \neq \pm\alpha^n,\ \pm\frac{3n+1}{n+1}\alpha^n & \text{for } n \text{ odd}, \\ \xi^n \neq \alpha^n,\ \frac{3n+1}{n+1}\alpha^n,\ (n+1)\alpha^n & \text{for } n \text{ even}. \end{cases}$$

4. Proofs for method C

The interpolation problem has been formulated in the introduction as:

- given polynomials $p(z)$ and $q(z)$ of degrees n and m,
- with simple zeros z_i, w_j, respectively, all different,
- find a polynomial $P(z)$ of degree at most $n+m-1$ with

$$P(z_i) = P(w_j) = 0,\ P''(z_i) = 0. \tag{9}$$

Because of the first two sets of conditions in (9), we can write

$$P(z) = p(z)q(z)Q(z),\ \text{degree}\, Q(z) \leq n-1. \tag{10}$$

The final conditions of (9) then lead to

$$2p'(z_i)q(z_i)Q'(z_i) + \{p''(z_i)q(z_i) + 2p'(z_i)q'(z_i)\}Q(z_i) = 0, \tag{11}$$

with z_i the n zeros of $p(z)$.

Proof of Theorem 2.1. Inserting (3) into (11) and using (4) we find

$$(\alpha_0 + \alpha_1 z_i)Q'(z_i) + \beta_0 Q(z_i) = 0,\ 1 \leq i \leq n. \tag{12}$$

Because of the degree restriction on Q, at most $n-1$, this immediately implies

$$(\alpha_0 + \alpha_1 z)Q'(z) + \beta_0 Q(z) = 0. \tag{13}$$

Solving this linear first order ordinary differential equation for the cases $\alpha_1 = 0$ (distinguishing $\alpha_0 = 0$ or $\alpha_0 \neq 0$) and $\alpha_1 \neq 0$, we find that $Q(z)$ has to be identically zero under the condition stated in the theorem ($\alpha_1 \neq 0$ was the only case that (13) really had a non-trivial polynomial solution of degree at most $n-1$; that is where $-\beta_0/\alpha_1 \notin \{1, 2, \ldots, n-1\}$ comes in). □

Proof of Theorem 2.2. Proceeding as in the previous proof, but now the degree of the polynomial on the left-hand side of the equation could be equal to the degree of $p(z)$,we arrive at the differential equation

$$(\alpha_0 + \alpha_1 z + \alpha_2 z^2)Q'(z) + (\beta_0 + \beta_1 z)Q(z) = Cp(z) \tag{14}$$

for the polynomial Q of degree at most $n-1$. The equation (14) can be solved with an integrating factor $\mu(z)$ following from

$$\frac{\mu'(z)}{\mu(z)} = \frac{\beta_0 + \beta_1 z}{\alpha_0 + \alpha_1 z + \alpha_2 z^2}$$

and we find

$$(\alpha_0 + \alpha_1 z + \alpha_2 z^2)Q(z) = C\int_{\sigma_1}^{z} (\zeta - \sigma_1)^{A-1}(\zeta - \sigma_2)^{B-1} p(\zeta)d\zeta + D. \tag{15}$$

Now the left-hand side has a zero for $z = \sigma_1$ and $z = \sigma_2$; the first gives $D = 0$ and the second, in view of the condition stated in (8), that $C = 0$. Thus $Q \equiv 0$, implying $P \equiv 0$. □

5. Proofs for the new regular problems

The five cases of theorem 3.1 will each be proved using each of the methods **A**, **B** and **C**.

5.1. Method A

Proof of 1. The interpolating polynomial has degree at most $2n$; write

$$P(z) = \sum_{k=0}^{n-1} a_k z^k + \sum_{k=0}^{n-1} b_k z^{n+k} + cz^{2n}.$$

This has to vanish at the n zeros z_j of $z^n - \alpha^n$:

$$\sum_{k=0}^{n-1} (a_k + \alpha^n b_k) z^k + c\alpha^{2n} = 0,$$

leading to a polynomial of degree at most $n-1$ having n zeros, thus:

$$a_0 + \alpha^n b_0 + \alpha^{2n} c_0 = 0, \tag{16a}$$

$$a_k + \alpha^n b_k = 0, 1 \leq k \leq n-1. \tag{16b}$$

The condition $P(0) = 0$ implies:

$$a_0 = 0. \tag{17}$$

As the derivative only has to be looked at in the points $z_j \neq 0$, we can as well put $z_j^2 P''(z_j) = 0$:

$$\sum_{k=0}^{n-1} \{k(k-1)a_k + (n+k)(n+k-1)\alpha^n b_k\} z^k + 2n(2n-1)\alpha^{2n} c = 0,$$

which implies

$$n(n-1)\alpha^n b_0 + 2n(2n-1)\alpha^n c_0 = 0, \tag{18a}$$

$$k(k-1)a_k + (n+k)(n+k-1)\alpha^n b_k = 0, 1 \leq k \leq n-1. \tag{18b}$$

The equations (16b) and (18b) immediately imply $a_k = b_k = 0$ for $1 \leq k \leq n-1$ (the determinant of the matrix for the 2×2 system for fixed k is $n(n+2k-1)\alpha^n \neq 0$).

Inserting (17) in (16a) and (18a) gives a 2×2 system for b_0, c_0 with determinant $n(3n-1)\alpha^{3n} \neq 0$, leading to $b_0 = c_0 = 0$ and thus $P \equiv 0$. □

Proof of 2. This time the interpolating polynomial has degree at most $3n-1$ and we write

$$P(z) = \sum_{k=0}^{n-1} \left(a_k + b_k z^n + c_k z^{2n}\right) z^k$$

This has to vanish at the zeros of both $z^n - \alpha^n$ and $z^n - \beta^n$; as in the previous proof, we find

$$a_k + \alpha^n b_k + \alpha^{2n} c_k = 0, \ 0 \leq k \leq n-1, \tag{19a}$$

$$a_k + \beta^n b_k + \beta^{2n} c_k = 0, \ 0 \leq k \leq n-1. \tag{19b}$$

Moreover, the second derivative of P has to vanish in the zeros of $z^n - \alpha^n$ and, looking at $z^2P''(z)$ as before, we find

$$k(k-1)a_k + (n+k)(n+k-1)\alpha^n b_k + (2n+k)(2n+k-1)\alpha^{2n}c_k = 0,\ 0 \le k \le n-1. \tag{20}$$

Combining (19) and (20), we see that the triple $\{a_k, b_k, c_k\}$ satisfies, for each fixed k from $\{0, 1, \ldots, n-1\}$ the linear system

$$\mathcal{A}_k \begin{pmatrix} a_k \\ b_k \\ c_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

with

$$\mathcal{A}_k = \begin{pmatrix} 1 & \alpha^n & \alpha^{2n} \\ 1 & \beta^n & \beta^{2n} \\ k(k-1) & (n+k)(n+k-1)\alpha^n & (2n+k)(2n+k-1)\alpha^2 n \end{pmatrix}.$$

As

$$\det\mathcal{A}_k = n\alpha^n(\beta^n - \alpha^n)\left[(3n+2k-1)\alpha^n - (n+2k-1)\beta^n\right] \neq 0,$$

we conclude that, given the conditions, the system has the trivial solution only. □

Proof of 3. In this case we incorporate the node generator $z^{kn} - \alpha^{kn}$ in the definition of the interpolating polynomial:

$$P(z) = \sum_{j=0}^{kn-1} \left(a_j + b_j z^{kn}\right) z^j$$

with $b_j = 0$ for $n \le j \le kn-1$. Using the method as in the previous cases, we find

$$a_j + \alpha^{kn} b_j = 0,\ 0 \le j \le kn-1.$$

Combining this with the known values for the b_j, this leads to $a_j = 0$ for $n \le j \le kn-1$; thus the interpolating polynomial reduces to

$$P(z) = \sum_{j=0}^{n-1} \left(a_j + b_j z^{kn}\right) z^j$$

with

$$a_j + \alpha^{kn} b_j = 0,\ 0 \le j \le n-1. \tag{21}$$

Just as in the previous cases, the conditions $z_j^2 P''(z_j) = 0$ give

$$j(j-1)a_j + (nk+j)(nk+j-1)\alpha^{kn} b_j = 0,\ 0 \le j \le n-1. \tag{22}$$

Combination of (21) and (22) leads, for each fixed j in the range, to a system with non-vanishing determinant [here $kn\alpha^{kn}(kn+2j-1)$]. □

Proof of 4. *and* 5. It is quite obvious that the method used in the previous proofs does not lead to a 'linear algebra' problem that can be managed so easily. □

5.2. Method B

Proof of 1. Put

$$P(z) = z(z^n - \alpha^n)Q(z)$$

with $\deg Q \leq n-1$. Inserting the zeros z_j of $z^n - \alpha^n$ in the second derivative, we find

$$0 = P''(z_j) = 2nz_j^n Q'(z_j) + n(n+1)z_j^{n-1}Q(z_j).$$

Dividing by $z_j^{n-1} \neq 0$, we see that the polynomial

$$zQ'(z) + \frac{n+1}{2}Q(z),$$

which is of degree at most $n-1$, has n zeros. Therefore it satisfies

$$zQ'(z) + \frac{n+1}{2}Q(z) = 0.$$

This differential equation has the solution $Cz^{-(n-1)/2}$: for $n \geq 2$ the only polynomial solution is the trivial one. Moreover, from the method of proof it is clear that in the case $n = 1$ we do not find a differential equation, but just $Q(\alpha) = 0$ with Q a constant: $Q \equiv 0$. □

Proof of 2. This time we put

$$P(z) = (z^n - \alpha^n)(z - \beta^n)Q(z) \text{ with } \deg Q \leq n-1.$$

Inserting the zeros z_j of $z^n - \alpha^n$ in the second derivative, we find

$$0 = P''(z_j) = 2nz_j^{n-1}Q'(z_j) + \{n(n-1)z_j^{n-2}(\alpha^n - \beta^n) + 2n^2 z_j^{n-2}\}Q(z_j),$$

and after simplification

$$z_jQ'(z_j) + \frac{(3n-1)\alpha^n - (n-1)\beta^n}{2(\alpha^n - \beta^n)}Q(z_j) = 0.$$

Dropping the index on z we again arrive at a polynomial having more zeros than its degree, showing that Q satisfies a simple linear, homogenous differential equation of order 1. The solution can be written as

$$Q(z) = Ce^{-\xi z},\ C \in \mathbb{C},\ \xi = \frac{(3n-1)\alpha^n - (n-1)\beta^n}{2(\alpha^n - \beta^n)}.$$

Under the conditions on α, β given in the theorem, Q never reduces to a polynomial of degree at most $n-1$: the problem is regular. □

Proof of 3. Now the interpolating polynomial is determined by

$$P(z) = (z^{kn} - \alpha^{kn})Q(z) \text{ with } \deg Q \leq n-1.$$

As in the previous proofs, the interpolation conditions for the second derivative lead to an ordinary differential equations for Q:

$$zQ'(z) + \frac{kn-1}{2}Q(z) = 0.$$

Just as in the proof for case **1.** this leads to $Q \equiv 0$. □

Proof of 4. Put

$$P(z) = z(z - z_0)(z^n - \alpha^n)(z^n - \frac{3n+1}{n+1}\alpha^n)Q(z)$$

with $\deg Q \leq n-1$. Proceeding as before, we find after simplification

$$(z_j - z_0)Q'(z_j) + Q(z_j) = 0,\ 1 \leq j \leq n.$$

Again the number of zeros implies a differential equation

$$(z - z_0)Q'(z) + Q(z) = 0$$

which can be integrated immediately: $(z - z_0)Q(z) = C$, showing that $Q \equiv 0$.

The conditions on α, z_0 ensure that $p(z)$ and $q(z)$ have simple zeros only. □

Proof of 5. Now

$$P(z) = z(z^2 - \xi^2)(z^n - \alpha^n)(z^n - \frac{3n+1}{n+1}\alpha^n)Q(z)$$

with $\deg Q \leq n-1$. Proceeding as in **4.** the differential equation for Q turns out to be

$$(z^2 - \xi^2)Q'(z) + 2zQ(z) = C(z^n - \alpha^n)$$

as the degree on the right-hand side is equal to the number of zero conditions.

Solving this equation we find

$$(z^2 - \xi^2)Q(z) = C\int_0^z (t(n - \alpha^n)dt + D = Cz\left(\frac{z^n}{n+1} - \alpha^n\right) + D.$$

As the left-hand side has a zero for $z = \pm\xi$, this leads to a 2×2 system for the unknown constants C, D with determinant

$$\xi\left[\{1 + (-1)^n\}\frac{\xi^n}{n+1} - 2\alpha^n\right]$$

and the conditions on α, ξ ensure that this system has the trivial solution only. □

5.3. Method C

Proof of 1. The choices

$$\alpha_0 = 0,\ \alpha_1 = 2n,\ \beta_0 = n(n+1),\ r_1(z) = -2nz,\ r_2(z) = -n(n+1),$$

leading to $g(z) = z^n + z^{n-1} - \alpha^n$ show that the conditions (3) and (4) are satisfied; thus the case $\alpha_0 = 0$, $\beta_0 \neq 0$ applies. □

Proof of 2. The case $n = 1$ can be resolved using

$$\alpha_0 = -2\beta,\ \alpha_1 = 2,\ \beta_0 = 2,\ r_1(z) = -2(z - \beta),\ r_2(z) = -2$$

leading to $g(z) = z - \alpha + 1$.

For $n \geq 2$ take:

$$\alpha_0 = 0,\ \alpha_1 = 2(\alpha^n - \beta^n),\ \beta_0 = (3n-1)\alpha^n - (n-1)\beta^n,$$
$$r_1(z) = 2nz^{n-1},\ r_2(z) = n(3n-1)z^{n-2}$$

leading to $g(z) = nz^{n-2}$. □

Proof of 3. Use

$$\alpha_0 = 0,\ \alpha_1 = 2,\ \beta_0 = nk - 1,\ g(z) = nkz^{nk-2}$$

and

$$r_1(z) = \left[2nz^{n-1}\frac{z^{kn} - \alpha^{kn}}{z^n - \alpha^n} - 2knz^{kn-1}\right] / (z^n - \alpha^n),$$

$$r_2(z) = \Bigg[n(n-1)z^{n-2}\frac{z^{kn} - \alpha^{kn}}{z^n - \alpha^n} + 2nz^{n-1}\times \\ \times\frac{\mathrm{d}}{\mathrm{dz}}\left(\frac{z^{kn} - \alpha^{kn}}{z^n - \alpha^n}\right) - kn(kn-1)z^{kn-2}\Bigg] / (z^n - \alpha^n).$$

The conditions are easily checked. □

Proof of 4. Choose

$$\alpha_0 + \alpha_1 z = z - z_0,\ \beta_0 = 1,\ r_1(z) = -2nz^n\frac{z - z_0}{z_0},\ r_2(z) = -2nz^{n-1}\frac{z - z_0}{z_0}.$$

The conditions follow from the requirements that p and q are co-prime and have simple zeros and from the condition (4) for

$$g(z) = nz\left\{(3n^2 + 4n + 1)z^{2n} - (3n^2 + 2n - 1)z_0 z^{2n-1} \right. \\ \left. - (3n^2 + 4n + 1)\alpha^n z^n + (3n^2 - 2n - 1)z_0\alpha^n z^{n-1}\right\} / ((n+1)z_0). \tag{23}$$

□

Proof of 5. The proof uses

$$\alpha_0 + \alpha_1 z + \alpha_2 z^2 = z^2 - \xi^2,\ \ \beta_0 + \beta_1 z = 2z,$$

$$r_1(z) = -n(3n+1)z^n(z^2 - \xi^2)^2/(2\xi^2),\ \ r_2(z) = -n(3n+1)z^{n-1}(z^2 - \xi^2)^2/\xi^2$$

and $g(z)$ given by

$$\frac{n[(3n^2 + 4n + 1)(z^n - \alpha^n)z^{n+2} + \xi^2 z^n\left\{(3n^2 - 8n - 3)\alpha^n - 3(n-1)z^n\right)\}]}{2(n+1)\xi^2}.$$

The conditions a.o come in to ensure that q has simple zeros. The condition (6) is automatically satisfied because $g(z_j) = -4n^2\alpha^{2n}/(n+1)$.

The condition (8), in the integral $A = B = 1$, is automatically satisfied for n odd and leads to $\xi^n \neq (n+1)\alpha^n$ for n even. □

Discussion

From the proofs it has become clear that method **A** necessitates a very special form for the node generating polynomials, while method **B** depends on the possibility of degree reduction in order to find a differential equation that can be solved without too many difficulties.

The fact that there does not appear to be much difference between the applicability of the methods **B** and **C** lies in the fact that both methods exploit the method of reducing the differential equation for the intermediary Q to a simple one.

The advantage of method **C** then lies in the fact that it enables one to find 'companion' node generating polynomials $q(z)$ to a *given* node generating polynomial $p(z)$ that lead to a regular $(0, 2)$ Pál-type interpolation problem on the nodes $\{p(z)q(z), p(z)\}$.

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